

# MAXIMAL SLICE IN ANTI-DE SITTER SPACE

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**ABSTRACT.** In this paper, we prove the existence of maximal slices in anti-de Sitter spaces (ADS spaces) with small boundary data at spatial infinity. The main argument is implicit function theorem. We also get a necessary and sufficient condition for boundary behavior of totally geodesic slice in ADS space. Moreover, we show that any isometric and maximal embedding of hyperbolic spaces into ADS space must be totally geodesic. Together with this, we see that most of maximal slices we get in this paper are not isometric to hyperbolic spaces, which implies that the Bernstein Theorem in ADS space fails.

## 1. INTRODUCTION

Finding a minimal surface with the given boundary data is an interesting problem in Riemannian geometry. Particularly, the existence and regularity of the minimal hypersurfaces with a prescribing asymptotic boundary at infinity in hyperbolic space  $\mathbf{H}^n$  have been discussed in [2], [3], [10], [11], etc. On the other hand, we know that a maximal slice, which is a spacelike hypersurface of a Lorentzian manifold and critical point of the induced area functional, plays an important role in General Relativity. It was used in the first proof of the positive mass theorem ([12]) and in the analysis of the Cauchy problem for asymptotically flat spacetimes. Many interesting results for the existence of compact maximal slice had been obtained in e.g. [4], [5], [7]. For complete noncompact case, we have known that there are entire solutions in asymptotically flat spacetime (see [4]). It should be pointed out that a complete maximal hypersurface in the Minkowski space must be totally geodesic, i.e. a hyperplane( see [6]). Anti-de Sitter(ADS) space is a Lorentzian manifold with negative constant sectional curvature, it plays the similar role in Lorentzian geometry as hyperbolic space does in Riemannian geometry. So, it is nature to study maximal slices in

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ADS spaces, this is one hand; on the other hand, we note that all the time slices ( level sets of the time function ) are isometric to the  $\mathbf{H}^n$  and are totally geodesic, hence are maximal. It may be of some interest in view of geometry to find some maximal slices which are not totally geodesic. By assuming a global barrier condition in asymptotically ADS space, K. Akutagawa proved the existence for the entire maximal slice with certain decay of height function at infinity in [1]; in ADS space case, he has also shown that, if the height function of the maximal slice satisfying this decay condition at spatial infinity, the maximal slice must be time slice (Proposition 3 in [1]).

In this paper, we obtain some maximal slices by implicit function theorem, which can be regarded as perturbations of time slices. These maximal slices are  $C^{1,1}$  up to the boundary. We also get a necessary and sufficient condition for boundary behavior of totally geodesic slice in ADS space. Moreover, we show that any isometric and maximal embedding of hyperbolic spaces into ADS space must be totally geodesic. Together with this, we see that most of maximal slices we get in this paper are not isometric to hyperbolic spaces, which implies that the Bernstein Theorem in ADS space fails.

Indeed, a maximal slice in ADS space satisfies a second order PDE in  $\mathbf{H}^n$ (see (1)). Therefore, it nature to consider the Dirichlet problem for the maximal slice of ADS space with infinity boundary value on  $\mathbf{H}^n$ . We shall address this problem in the forthcoming papers.

This paper is organized as follows: In Section 2, we derive the equation satisfied by the maximal slices and its corresponding linearized equation. In Section 3, we show that the linearized operator is an isomorphism between some weighted Hölder spaces. Hence, using the implicit function theorem, we prove our main result Theorem 3.1. In Section 4, we prove a necessary and sufficient condition for the boundary behavior of totally geodesic slice in ADS space, we also show that isometric and maximal embedding of  $\mathbf{H}^n$  into ADS spaces is totally geodesic, by these facts we see that most of our solutions are not totally geodesic.

## 2. MAXIMAL SLICE EQUATION IN ANTI-DE SITTER SPACE

In this section, we will derive the maximal slice equation in anti-de Sitter space. Let us begin with some basic facts. Suppose  $\mathbf{H}^n = (\mathbf{R}^+ \times \mathbf{S}^{n-1}, \sinh^{-2} \rho(d\rho^2 + d\sigma_0^2))$ , where  $d\sigma_0^2$  is the standard metric on  $\mathbf{S}^{n-1}$ . Then  $n + 1$  dimensional anti-de Sitter space  $V$  can be expressed as a warp product of  $\mathbf{R}$  and  $\mathbf{H}^n$ , namely,  $V = (\mathbf{R} \times \mathbf{H}^n, ds^2)$ , here  $ds^2 = -\coth^2 \rho dt^2 + \sinh^{-2} \rho(d\rho^2 + d\sigma_0^2)$ . As well known,  $V$  is a

vacuum solution of Einstein fields equations with negative cosmological constant. We denote the canonical connection in  $V$  by  $\overline{\nabla}$ . Let  $M^n$  be a smooth spacelike hypersurface in  $V$ . The height function  $u \in C^\infty(M)$  of  $M$  is the restriction of the time function  $t$  to  $M$ , then  $M$  can be regarded as a graph over  $\mathbf{H}^n$ , in the following, we assume  $M = \{(x, u(x)) | x \in \mathbf{H}^n\}$ , and  $u$  is defined on the whole  $V$  by requiring  $\frac{\partial}{\partial t}u = 0$ .

Note that  $M$  is then a level set of  $f(t, x) = t - u(x)$ , by a direct computation, we see that the future-directed unit normal vector  $N$  to  $M$  is

$$N = |\overline{\nabla}f|^{-1}[\overline{\nabla}f] = \frac{1}{\sqrt{1 - \coth^2 \rho |\nabla u|^2}}(\coth \rho \nabla u + \tanh \rho \frac{\partial}{\partial t}),$$

here and in the sequel,  $\nabla$ ,  $div$ , and  $\Delta$  are the gradient, divergence and Laplacian operator on  $\mathbf{H}^n$  respectively.

Let  $\square$  be the wave operator in  $V$ ,  $H_M$  be the mean curvature of  $M$  in  $V$  with respect to  $N$ , then, by a direct computation, we see that

$$\square f = -\Delta u + \tanh \rho \frac{\partial u}{\partial \rho}.$$

On the other hand, we also have

$$\square f = -NNf - H_M \cdot Nf,$$

thus, we see that

$$H_M = \tanh \rho \operatorname{div}\left(\frac{\coth^2 \rho \nabla u}{\sqrt{1 - \coth^2 \rho |\nabla u|^2}}\right)$$

If  $M$  is maximum, we have

$$\tanh \rho \operatorname{div}\left(\frac{\coth^2 \rho \nabla u}{\sqrt{1 - \coth^2 \rho |\nabla u|^2}}\right) = 0.$$

One can easily verify that the above equation is equivalent to

$$(1) \quad \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - \coth^2 \rho |\nabla u|^2}}\right) - \frac{2 \tanh \rho \frac{\partial u}{\partial \rho}}{\sqrt{1 - \coth^2 \rho |\nabla u|^2}} = 0.$$

By the structure of the equation, we find that if  $u_\epsilon$  is the solution of the following equation

$$(2) \quad \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - \epsilon \coth^2 \rho |\nabla u|^2}}\right) - \frac{2 \tanh \rho \frac{\partial u}{\partial \rho}}{\sqrt{1 - \epsilon \coth^2 \rho |\nabla u|^2}} = 0,$$

for some  $\epsilon > 0$ , then  $\sqrt{\epsilon}u_\epsilon$  is the solution for equation (1).

In the following, we consider a family of operators:

$$\begin{aligned} F(u, \epsilon) &:= \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - \epsilon \coth^2 \rho |\nabla u|^2}}\right) - \frac{2 \tanh \rho \frac{\partial u}{\partial \rho}}{\sqrt{1 - \epsilon \coth^2 \rho |\nabla u|^2}} \\ &= 0, \end{aligned}$$

and it is easy to see that

$$F(u, 0) = \Delta u - 2 \tanh \rho \frac{\partial u}{\partial \rho}$$

is the linearize equation of (1) at its trivial solution  $u = 0$ .

For the purpose of further discussion, we need to consider the following Dirichlet problem

$$(3) \quad \begin{cases} \mathbf{L}(u) := \Delta u - 2 \tanh \rho \frac{\partial u}{\partial \rho} = 0, & \text{in } \mathbf{H}^n \\ u|_{\mathbf{S}^{n-1}} = \varphi \end{cases}$$

where  $\varphi$  is a smooth function defined on the infinity boundary of  $\mathbf{H}^n$ . In (3) and the sequel,  $\mathbf{S}^{n-1}$  is regarded as the infinity boundary of  $\mathbf{H}^n$ . Besides above facts, we need to introduce the ball model for  $\mathbf{H}^n$  which is denoted by  $(\mathbf{D}^n, dS^2)$ , here,  $\mathbf{D}^n$  is the unit ball in  $\mathbf{R}^n$ , and  $dS^2$  is the standard hyperbolic metric which is defined as following:

$$dS^2 = \tau^{-2} \sum_{i=1}^n (dx^i)^2,$$

where  $\tau(x) = \frac{1}{2}(1 - |x|^2)$  and  $\sum_{i=1}^n (dx^i)^2$  is the Euclidean metric. The

relation between  $\rho$  and  $\tau$  can be expressed as  $\sinh \rho = \frac{r}{\tau}$ , where  $r(x) = |x|$  is the Euclidean distance from the origin. Hence the equation (1), (2) and (3) can be written as

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - (\frac{1-\tau}{\tau})^2 |\nabla u|^2}}\right) + \frac{2\tau}{1-\tau} \frac{\sum_{i=1}^n x^i \frac{\partial u}{\partial x^i}}{\sqrt{1 - (\frac{1-\tau}{\tau})^2 |\nabla u|^2}} = 0,$$

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - \epsilon\left(\frac{1-\tau}{\tau}\right)^2|\nabla u|^2}}\right) + \frac{2\tau}{1-\tau} \frac{\sum_{i=1}^n x^i \frac{\partial u}{\partial x^i}}{\sqrt{1 - \epsilon\left(\frac{1-\tau}{\tau}\right)^2|\nabla u|^2}} = 0.$$

and

$$(4) \quad \begin{cases} \mathbf{L}(u) := \Delta u + \frac{2\tau}{1-\tau} \sum_{i=1}^n x^i \frac{\partial u}{\partial x^i} = 0 & \text{in } \mathbf{H}^n \\ u|_{\mathbf{S}^{n-1}} = \varphi \end{cases}$$

respectively.

### 3. EXISTENCE OF MAXIMAL SLICE, WEIGHTED HÖLDER SPACE AND ANALYSIS OF LINEARIZE EQUATION

In this section, we will prove the existence of maximal slices in  $V$  with certain boundary data at infinity. More specifically, we are going to show

**Theorem 3.1.** *For any  $\varphi \in C^{4,\alpha}(\mathbf{S}^{n-1})$ , there is a  $\delta = \delta(\varphi) > 0$  so that for any  $\epsilon \in (0, \delta)$ , the following Dirichlet problem*

$$(5) \quad \begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - \epsilon\left(\frac{1-\tau}{\tau}\right)^2|\nabla u|^2}}\right) + \frac{2\tau}{1-\tau} \frac{\sum_{i=1}^n x^i \frac{\partial u}{\partial x^i}}{\sqrt{1 - \epsilon\left(\frac{1-\tau}{\tau}\right)^2|\nabla u|^2}} = 0, & \text{in } \mathbf{H}^n \\ u|_{\mathbf{S}^{n-1}} = \sqrt{\epsilon}\varphi \end{cases}$$

*admits a solution  $u \in C^2(\mathbf{D}^n)$  with  $\|u\|_{C^2(\mathbf{D}^n)} \leq C$ , here  $C$  is a constant depends on  $\varphi$ .*

**Remark 3.2.** (1) *In Theorem 3.1, we adopt the ball model for  $\mathbf{H}^n$ , and  $u$  is regarded as a function defined on  $\mathbf{D}^n$ .*

(2) *The second fundamental form of the solution we get in Theorem 3.1 decays as  $O(\tau^2)$  as  $\tau$  goes to 0. And we conjecture that the solution with the second fundamental form faster than quadratic decay must be  $\mathbf{H}^n$ .*

In order to prove Theorem 3.1, we will get some basic estimates of the linear equation by which we are able to show the corresponding linear elliptic operator is a linear isomorphism between some function spaces. To do this, let us first introduce a kind of weighted Hölder spaces (for more details, please see [8].). In the following, we will define weighted Hölder spaces on  $\Omega \subset \mathbf{D}^n$ , for  $0 \leq k \in \mathbf{Z}$  let  $C^k(\overline{\Omega})$  be the usual Banach spaces of  $k$  times continuously differential functions on  $\overline{\Omega}$ , and

$0 < \alpha < 1$  denote by  $C^{k,\alpha}(\overline{\Omega})$  the subspace of functions whose  $k$ -th derivatives satisfy a uniform Hölder condition of order  $\alpha$ , with the usual norms denoted by  $\|\cdot\|_{k;\Omega}$ ,  $\|\cdot\|_{k,\alpha;\Omega}$  respectively. And denote  $C^k(\Omega)$  and  $C^{k,\alpha}(\Omega)$  the linear space of functions satisfying the corresponding estimates uniformly on compact subsets of  $\Omega$ . For  $s \in \mathbf{R}$  define

$$\|w\|_{k,0;\Omega}^{(s)} = \sum_{l=0}^k \sum_{|\gamma|=l} \|\tau^{-s+l} \partial^\gamma w\|_{L^\infty(\Omega)},$$

where for any multi-index  $\gamma$ ,  $\partial^\gamma = \frac{\partial^{|\gamma|}}{\partial x^\gamma}$ ; and for  $0 < \alpha < 1$  define

$$\begin{aligned} \|w\|_{k,\alpha;\Omega}^{(s)} &= \|w\|_{k,0;\Omega}^{(s)} \\ &+ \sum_{|\gamma|=k} \sup_{x,y \in \Omega} [\min(\tau^{-s+k+\alpha}(x), \tau^{-s+k+\alpha}(y)) \frac{|\partial^\gamma w(x) - \partial^\gamma w(y)|}{|x-y|^\alpha}] \end{aligned}$$

Let  $\Lambda_{k,\alpha;\Omega}^s = \{w \in C^{k,\alpha}(\Omega) \mid \|w\|_{k,\alpha;\Omega}^{(s)} < +\infty\}$ , which is a Banach space. For  $x \in \mathbf{H}^n$ , let  $B(x)$  be the open Euclidean ball with center  $x$  and radius  $\frac{1}{3}\tau(x)$ . It is clear that

**Lemma 3.3.** *For any  $\Omega' \subset \Omega \subset \mathbf{H}^n$ , we have  $\Lambda_{k,\alpha;\Omega}^s \subset \Lambda_{k,\alpha;\Omega'}^s$ , and*

$$\|w\|_{k,\alpha;\Omega'}^{(s)} \leq \|w\|_{k,\alpha;\Omega}^{(s)}$$

for any  $w \in \Lambda_{k,\alpha;\Omega}^s$ ; also, For any  $\Omega_m \subset \Omega_{m+1} \subset \mathbf{H}^n$  and  $\bigcup_m \Omega_m = \mathbf{H}^n$ , we have

$$\|w\|_{k,\alpha;\mathbf{H}^n}^{(s)} \leq \sup_m \|w\|_{k,\alpha;\Omega_m}^{(s)}$$

for any  $w \in \Lambda_{k,\alpha;\mathbf{H}^n}^s$ .

*Proof.* By the definition, we see that for any  $w \in \Lambda_{k,\alpha;\Omega}^s$ ,

$$\|\tau^{-s+l} \partial^\gamma w\|_{L^\infty(\Omega')} \leq \|\tau^{-s+l} \partial^\gamma w\|_{L^\infty(\Omega)},$$

and

$$\begin{aligned} &\sup_{x,y \in \Omega'} [\min(\tau^{-s+k+\alpha}(x), \tau^{-s+k+\alpha}(y)) \frac{|\partial^\gamma w(x) - \partial^\gamma w(y)|}{|x-y|^\alpha}] \\ &\leq \sup_{x,y \in \Omega} [\min(\tau^{-s+k+\alpha}(x), \tau^{-s+k+\alpha}(y)) \frac{|\partial^\gamma w(x) - \partial^\gamma w(y)|}{|x-y|^\alpha}] \end{aligned}$$

Thus, we see that

$$||w||_{k,\alpha;\Omega'}^{(s)} \leq ||w||_{k,\alpha;\Omega}^{(s)};$$

and thus  $\Lambda_{k,\alpha;\Omega'}^s \subset \Lambda_{k,\alpha;\Omega}^s$ . On the other hand, for any  $\epsilon > 0$ , there is an  $x \in \mathbf{H}^n$ , we may assume  $x \in \Omega_m$  so that

$$|\tau^{-s+l}\partial^\gamma w(x)| > \|\tau^{-s+l}\partial^\gamma w\|_{L^\infty(\mathbf{H}^n)} - \epsilon,$$

and

$$|\tau^{-s+l}\partial^\gamma w(x)| \leq \|\tau^{-s+l}\partial^\gamma w\|_{L^\infty(\Omega_m)}$$

hence, we see

$$\|\tau^{-s+l}\partial^\gamma w\|_{L^\infty(\mathbf{H}^n)} - \epsilon \leq \|\tau^{-s+l}\partial^\gamma w\|_{L^\infty(\Omega_m)}$$

By the same arguments, we have

$$\begin{aligned} & \sup_{x,y \in \mathbf{H}^n} [\min(\tau^{-s+k+\alpha}(x), \tau^{-s+k+\alpha}(y)) \frac{|\partial^\gamma w(x) - \partial^\gamma w(y)|}{|x-y|^\alpha}] \\ & \leq \sup_{x,y \in \Omega_m} [\min(\tau^{-s+k+\alpha}(x), \tau^{-s+k+\alpha}(y)) \frac{|\partial^\gamma w(x) - \partial^\gamma w(y)|}{|x-y|^\alpha}] + \epsilon \end{aligned}$$

Thus, for any  $\epsilon > 0$ , and sufficiently large  $m$ , we have

$$||w||_{k,\alpha;\mathbf{H}^n}^{(s)} \leq \sup_m ||w||_{k,\alpha;\Omega_m}^{(s)} + \epsilon$$

which implies the conclusion is true.  $\square$

The following lemma is the same as Lemma 3.1 in [8].

**Lemma 3.4.** *For  $x \in \Omega$ , then*

$$||w||_{k,\alpha;B(x) \cap \Omega}^{(s)} \leq ||w||_{k,\alpha;\Omega}^{(s)}$$

and,

$$||w||_{k,\alpha;\Omega}^{(s)} \leq C \sup_{x \in \Omega} ||w||_{k,\alpha;B(x) \cap \Omega}^{(s)}$$

where  $C$  depends only on  $k$ .

Let  $B(0)$  denote the open Euclidean ball with center 0 and radius  $\frac{1}{3}$ , and for  $x \in \mathbf{H}^n$  define  $\psi_x : B(0) \rightarrow B(x)$  by

$$(6) \quad y := \psi_x(z) = x + \tau(x)z.$$

If  $y \in B(x)$ , then

$$(7) \quad \frac{1}{10}\tau(x) \leq \tau(y) \leq 40\tau(x).$$

Therefore, there exist a universal constant  $\Lambda_1$  such that

$$\begin{aligned} \Lambda_1^{-1} \tau^{-s+l}(x) \|\partial^\gamma w\|_{L^\infty(B(x) \cap \Omega)} &\leq \|\tau^{-s+l} \partial^\gamma w\|_{L^\infty(B(x) \cap \Omega)} \\ &\leq \Lambda_1 \tau^{-s+l}(x) \|\partial^\gamma w\|_{L^\infty(B(x) \cap \Omega)} \end{aligned}$$

where  $\Lambda_1$  depends only on  $s$  and  $l$ . Let  $v(z) = w \circ \psi_x(z)$ , we have  $\frac{\partial^\gamma v}{\partial z^\gamma} = \tau^l(x) \frac{\partial^\gamma w}{\partial y^\gamma}$  for  $|\gamma| = l$ . So for any  $y \in B(x) \cap \Omega$ , we can show

$$\tau^{-s+l}(x) \partial^\gamma w(y) = \tau^{-s}(x) \partial^\gamma v(z).$$

Using (6) and (7), one can conclude that

$$\begin{aligned} \Lambda_1^{-1} \tau^{-s}(x) \|\partial^\gamma v\|_{L^\infty(\psi_x^{-1}(B(x) \cap \Omega))} &\leq \|\tau^{-s+l} \partial^\gamma w\|_{L^\infty(B(x) \cap \Omega)} \\ &\leq \Lambda_1 \tau^{-s}(x) \|\partial^\gamma v\|_{L^\infty(\psi_x^{-1}(B(x) \cap \Omega))} \end{aligned}$$

By this, it follows that

$$(8) \quad \begin{aligned} \Lambda^{-1} \tau^{-s}(x) \|v\|_{k, \alpha; \psi_x^{-1}(B(x) \cap \Omega)} &\leq \|w\|_{k, \alpha; B(x) \cap \Omega}^{(s)} \\ &\leq \Lambda \tau^{-s}(x) \|v\|_{k, \alpha; \psi_x^{-1}(B(x) \cap \Omega)} \end{aligned}$$

where  $\Lambda$  is only depended on  $k$ ,  $\alpha$  and  $s$ .

Next, consider the following

$$(9) \quad \begin{cases} \mathbf{L}(u) = \Delta u - 2 \tanh \rho \frac{\partial u}{\partial \rho} = \Delta u + 2 \frac{\tau(y)}{1-\tau(y)} \sum_{i=1}^n y^i \frac{\partial}{\partial y^i} u = \eta \\ u|_{\mathbf{S}^{n-1}} = 0 \end{cases}$$

here,  $\eta \in \Lambda_{0, \alpha; \mathbf{H}^n}^s$  where  $s$  is to be determined later.

**Lemma 3.5.** *Suppose  $u \in C^2(\mathbf{H}^n) \cap \Lambda_{0,0; \mathbf{H}^n}^s$  is a solution for (9),  $\eta \in \Lambda_{k, \alpha; \mathbf{H}^n}^s$ . Then*

$$\|u\|_{k+2, \alpha; \mathbf{H}^n}^{(s)} \leq C(\|\eta\|_{k, \alpha; \mathbf{H}^n}^{(s)} + \|u\|_{0,0; \mathbf{H}^n}^{(s)})$$

here  $C = C(k, \alpha)$ .

*Proof.* It is easy to see that (9) is equivalent to

$$(10) \quad \begin{cases} \tau^2(y) \Delta_0 u + \tau(y)(n-2 + \frac{2}{1-\tau(y)}) \sum_{i=1}^n y^i \frac{\partial}{\partial y^i} u = \eta \quad \text{in } \mathbf{D}^n \\ u|_{\mathbf{S}^{n-1}} = 0, \end{cases}$$

where  $\Delta_0$  is the standard Laplacian for  $\mathbf{D}^n \subset \mathbf{R}^n$ . Suppose  $v(z) = u \circ \psi_x(z)$  for  $\forall z \in B(0)$ , then (10) becomes



$$(11) \quad \begin{cases} \frac{\tau^2(y)}{\tau^2(x)} \Delta_0 v + \frac{\tau(y)}{\tau(x)} (n-2 + \frac{2}{1-\tau(y)}) \sum_{i=1}^n y^i \frac{\partial}{\partial z^i} v = \eta & \text{in } B(0) \\ u|_{\mathbf{S}^{n-1}} = 0. \end{cases}$$

Let  $B'(0)$  and  $B'(x)$  denote the open Euclidean balls with center 0 and radius  $\frac{1}{4}$  and with center  $x$  and radius  $\frac{1}{4}\tau(x)$  respectively. Since  $\frac{1}{100} \leq \frac{\tau^2(y)}{\tau^2(x)} \leq 160$  when  $y \in B(x)$  and  $n \leq n-2 + \frac{2}{1-\tau(y)} \leq n+2$ , it follows that (11) is uniformly elliptic equation on  $B(0)$ . Hence by standard Schauder theory, we have

$$\|v\|_{k+2,\alpha;B'(0)} \leq C(\|\eta \circ \psi_x\|_{k,\alpha;B(0)} + \|v\|_{0,0;B(0)})$$

where  $C$  only depends on  $k, \alpha$ . Choose  $\Omega' \subset \Omega$  such that  $B(x) \subset \Omega$  for any  $x \in \Omega'$ . Applying (8) and Lemma 3.4, we obtain

$$\begin{aligned} \|u\|_{k+2,\alpha;\Omega'}^{(s)} &\leq C \sup_{x \in \Omega'} \tau^{-s}(x) \|u \circ \psi_x\|_{k+2,\alpha;\psi_x^{-1}(B'(x) \cap \Omega')} \\ &\leq C \sup_{x \in \Omega'} \tau^{-s}(x) \|u \circ \psi_x\|_{k+2,\alpha;B'(0)} \\ &\leq C \sup_{x \in \Omega'} \tau^{-s}(x) (\|\eta \circ \psi_x\|_{k,\alpha;B(0)} + \|v\|_{0,0;B(0)}) \\ &\leq C(\|\eta\|_{k,\alpha;\Omega}^{(s)} + \|u\|_{0,0;\Omega}^{(s)}) \\ &\leq C(\|\eta\|_{k,\alpha;\mathbf{H}^n}^{(s)} + \|u\|_{0,0;\mathbf{H}^n}^{(s)}) \end{aligned}$$

Therefore lemma follows from Lemma 3.3.  $\square$

**Proposition 3.6.** *For any  $\eta \in \Lambda_{0,\alpha;\mathbf{H}^n}^s$ , there exists  $u \in C^2(\mathbf{H}^n) \cap \Lambda_{0,0;\mathbf{H}^n}^s$  for  $0 \leq s < n+1$  satisfying (9), and  $\|u\|_{0,0;\mathbf{H}^n}^{(s)} \leq C\|\eta\|_{0,\alpha;\mathbf{H}^n}^{(s)}$  with  $C$  depending on  $s$ . Moreover,  $u \in \Lambda_{2,\alpha;\mathbf{H}^n}^s$  with*

$$\|u\|_{2,\alpha;\mathbf{H}^n}^{(s)} \leq C\|\eta\|_{0,\alpha;\mathbf{H}^n}^{(s)},$$

here  $C$  is a constant depends only on  $s$  and  $\alpha$ .

*Proof.* Let  $\{\Omega_m\}_{m=1}^\infty$  be an exhausting sequence such that  $\Omega_m \subset \Omega_{m+1}$  and  $\bigcup_m \Omega_m = \mathbf{H}^n$ . Let  $w_m$  be the solution for the following equation:

$$\begin{cases} \mathbf{L}(w_m) = \eta & \text{in } \Omega_m \\ w_m|_{\partial\Omega_m} = 0. \end{cases}$$

Hence  $w_m \in C^{2,\alpha}(\overline{\Omega}_m)$  since  $\eta \in C^{0,\alpha}(\overline{\Omega}_m)$  ([8]).

Set  $\phi = \tau^s$ ,

$$\begin{aligned} \mathbf{L}(\phi) &= -s(2s-n+2)\tau^{s+1} + s(s-n-1)\tau^s + \frac{2}{1-\tau}s\tau^{s+1} \\ &\leq -s(2s-n-2)\tau^{s+1} + s(s-n-1)\tau^s \end{aligned}$$

since  $\frac{2}{1-\tau} \leq 4$  and  $s > 0$ . For  $0 \leq s < n+1$ , it is easy to check that

$$\mathbf{L}(\phi) = \Delta\phi - \frac{2\tau(y)(1-2\tau(y))}{1-\tau(y)} \frac{\partial}{\partial\tau}\phi \leq -\delta\phi$$

for some constant  $\delta > 0$  only depended on  $s$ . On the other hand, we have  $|\eta| \leq C\tau^s$  where  $C = \|\eta\|_{0,\alpha;\mathbf{H}^n}^{(s)}$  since  $\eta \in \Lambda_{0,\alpha;\mathbf{H}^n}^s$ . We choose a constant  $C_1 = C/\delta$  such that

$$\begin{cases} \mathbf{L}(w_m) \geq \mathbf{L}(C_1\phi) & \text{in } \Omega_m \\ (C_1\phi - w_m)|_{\partial\Omega_m} \geq 0 \end{cases}$$

By maximum principle,  $w_m \leq C_1\tau^s$ . By the same arguments, we may get the lower bound of  $w_m$ , hence,  $|w_m| \leq C_1\tau^s$ . Therefore  $w_m$  convergence to a function  $u \in C^2(\mathbf{H}^n) \cap \Lambda_{0,0;\mathbf{H}^n}^s$  which solves (9), and we have  $\|u\|_{0,0;\mathbf{H}^n}^{(s)} \leq C\|\eta\|_{0,\alpha;\mathbf{H}^n}^{(s)}$  where  $C$  depends only on  $s$ . By Lemma 3.5, we know  $u \in \Lambda_{2,\alpha;\mathbf{H}^n}^s$  with

$$\|u\|_{2,\alpha;\mathbf{H}^n}^{(s)} \leq C\|\eta\|_{0,\alpha;\mathbf{H}^n}^{(s)},$$

where  $C = C(s, \alpha)$ . □

Now by the Lemma 3.5 and Proposition 3.6, We have the following:

**Theorem 3.7.** *The operator  $\mathbf{L} : \Lambda_{2,\alpha;\mathbf{H}^n}^s \rightarrow \Lambda_{0,\alpha;\mathbf{H}^n}^s$  defined in (9) is an isomorphism, where  $0 \leq s < n+1$ .*

**Corollary 3.8.** *For any  $\varphi \in C^{4,\alpha}(\mathbf{S}^{n-1})$ , Dirichlet problem (3)(or (4)) has a solution.*

*Proof.* We use the cylindrical coordinate system  $(\rho, \theta)$ . And extend  $\varphi$  as  $\varphi(\rho, \theta) = \varphi(\theta)$ , for  $\theta \in \mathbf{S}^{n-1}$  and small  $\rho$ . Then let  $f(\rho, \theta) \in C^{2,\alpha}(\mathbf{H}^n)$  such that for some small  $\rho$ ,  $f(\rho, \theta) = \varphi + \frac{1}{2(n-1)}\rho^2\Delta_{\mathbf{S}^{n-1}}\varphi$ , here,  $\Delta_{\mathbf{S}^{n-1}}$  is the Laplacian operator on  $\mathbf{S}^{n-1}$ . Put  $f$  into left side of (3), one can see that

$$\begin{aligned} \mathbf{L}(f) &= \frac{1}{n-1}\sinh^2\rho\Delta_{\mathbf{S}^{n-1}}\varphi - \frac{1}{n-1}((n-2)\sinh\rho\cosh\rho + 2\tanh\rho) \\ &\quad \cdot \rho\Delta_{\mathbf{S}^{n-1}}\varphi + \sinh^2\rho\Delta_{\mathbf{S}^{n-1}}\varphi + \frac{1}{2(n-2)}\rho^2\sinh^2\rho\Delta_{\mathbf{S}^{n-1}}^2\varphi \\ &= O(\rho^4) = O(\tau^4) \quad \text{as } \tau \rightarrow 0. \end{aligned}$$

Because  $\mathbf{L}(f)$  is  $C^{0,\alpha}$  in any compact subset and behave like  $\tau^4$  near boundary, we conclude that  $\mathbf{L}(f) \in \Lambda_{0,\alpha;\mathbf{H}^n}^s$ , for any  $s \leq 4$ . Then the corollary follows from the Proposition 3.6. □

Now we are in the position to prove Theorem 3.1. By Corollary 3.8, (4) has a solution  $u$  satisfying  $u - f \in \Lambda_{2,\alpha;\mathbf{H}^n}^s$  for some  $s \in [0, 4]$  where

$f$  is given as in the proof of Corollary 3.8. Since (4) is a linear equation, we can multiply  $\varphi$  by a suitable constant such that the corresponding solution  $u$  satisfying  $v = \frac{1}{\sqrt{1 - (\frac{1-\tau}{\tau})^2 |\nabla u|^2}} < +\infty$ , that is,  $u$  is spacelike.

And define

$$\Xi_A = \{w \in \Lambda_{2,\alpha;\mathbf{H}^n}^2 \mid \frac{1}{\sqrt{1 - (\frac{1-\tau}{\tau})^2 |\nabla(w+u)|^2}} < A < +\infty\} \subset \Lambda_{2,\alpha;\mathbf{H}^n}^2$$

Obviously  $\Xi_A$  is nonempty open set of  $\Lambda_{2,\alpha;\mathbf{H}^n}^2$ , since  $0 \in \Xi_A$ . Define an operator

$$H(\cdot, \cdot) : (-1, +1) \times \Xi_A \rightarrow \Lambda_{0,\alpha;\mathbf{H}^n}^2$$

by

$$\begin{aligned} H(\epsilon, w) &:= \operatorname{div}\left(\frac{\nabla(w+u)}{\sqrt{1 - \epsilon(\frac{1-\tau}{\tau})^2 |\nabla(w+u)|^2}}\right) \\ &+ \frac{2\tau}{1-\tau} \frac{\sum_{i=1}^n x^i \frac{\partial(w+u)}{\partial x^i}}{\sqrt{1 - \epsilon(\frac{1-\tau}{\tau})^2 |\nabla(w+u)|^2}} \\ &= \frac{\Delta(w+u)}{\sqrt{1 - \epsilon(\frac{1-\tau}{\tau})^2 |\nabla(w+u)|^2}} \\ &+ \left\langle \nabla \frac{1}{\sqrt{1 - \epsilon(\frac{1-\tau}{\tau})^2 |\nabla(w+u)|^2}}, \nabla(w+u) \right\rangle \\ &+ \frac{2\tau}{1-\tau} \frac{\sum_{i=1}^n x^i \frac{\partial(w+u)}{\partial x^i}}{\sqrt{1 - \epsilon(\frac{1-\tau}{\tau})^2 |\nabla(w+u)|^2}} \end{aligned}$$

From Corollary 3.8, we have  $H(0, 0) = 0$ . By a direct computation, we see that  $H$  is a smooth operator, and for any  $h \in \Lambda_{2,\alpha;\mathbf{H}^n}^2$ ,

$$\frac{\partial}{\partial t} H(0, th)|_{t=0} = \Delta h + 2 \frac{\tau(y)}{1-\tau(y)} \sum_{i=1}^n x^i \frac{\partial}{\partial x^i} h$$

It follows that the map  $\frac{\partial}{\partial t} H(0, th)|_{t=0} = \mathbf{L} : \Lambda_{2,\alpha;\mathbf{H}^n}^2 \rightarrow \Lambda_{0,\alpha;\mathbf{H}^n}^2$  is an isomorphism from Theorem 3.7. Now by the implicit function theorem(cf. [9]), we can conclude that (5) has a solution whose difference

by  $u$  is in  $\Lambda_{2,\alpha;\mathbf{H}^n}^2$  and boundary data is given by small  $\sqrt{\epsilon}\varphi$ . Thus we finish to prove Theorem 3.1.

#### 4. BOUNDARY BEHAVIOR OF TOTALLY GEODESIC SLICE OF ADS SPACE

In this section, we will show that any isometric and maximal embedding of  $\mathbf{H}^n$  into ADS space is totally geodesic, and moreover, we will give a sufficient and necessary condition of the boundary value of the height function for totally geodesic slice. Together with Theorem 3.1, we know that the Bernstein Theorem in ADS spacetime fails. Let's begin with the following

**Proposition 4.1.** *If a hyperbolic space is isometrically immersed in the anti-de Sitter space as its maximal hypersurface, it must be totally geodesic.*

*Proof.* Suppose that  $M$  is a hyperbolic space, which is also a maximal hypersurface of anti-de Sitter space  $V$ . We choose a local field of Lorentzian orthonormal frames  $e_0, e_1, \dots, e_n$  in  $V$  such that, at each point of  $M$ ,  $e_1, \dots, e_n$  spans the tangent space of  $M$  and  $e_0$  is the unit timelike normal vector for  $M$ . We make use of the following convention on the ranges of indices:

$$0 \leq \alpha, \beta, \gamma, \dots \leq n, \quad 1 \leq i, j, k, \dots \leq n.$$

Let  $\omega_0, \omega_1, \dots, \omega_n$  be the dual frame field. Then the structure equations of  $V$  are

$$\begin{cases} d\omega_0 &= -\omega_{0i} \wedge \omega_i \\ d\omega_i &= \omega_{i0} \wedge \omega_0 - \omega_{ik} \wedge \omega_k \quad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0 \\ d\omega_{0i} &= -\omega_{0k} \wedge \omega_{ki} - K_{0i0j} \omega_0 \wedge \omega_j + \frac{1}{2} K_{0ijk} \omega_j \wedge \omega_k \\ d\omega_{ij} &= \omega_{i0} \wedge \omega_{0j} - \omega_{ik} \wedge \omega_{kj} - K_{ijok} \omega_0 \wedge \omega_k + \frac{1}{2} K_{ijkl} \omega_k \wedge \omega_l \end{cases}$$

where  $K_{\alpha\beta\gamma\delta}$  is the curvature tensor for  $V$ . We restrict these forms to  $M$ , then  $\omega_0 = 0$ . We may put  $\omega_{0i} = h_{ij}\omega_j$ , where  $h_{ij}$  is the components of the second fundamental form of  $M$ . And we also have the structure equation for  $M$ :

$$\begin{cases} d\omega_i &= -\omega_{ik} \wedge \omega_k \quad \omega_{ij} + \omega_{ji} = 0 \\ d\omega_{ij} &= -\omega_{ik} \wedge \omega_{kj} + \frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l \end{cases}$$

where  $R_{ijkl}$  is the curvature tensor of  $M$ . Hence we have the Gauss formula,

$$R_{ijkl} = K_{ijkl} - h_{ik}h_{jl} + h_{il}h_{jk}.$$

Since both  $V$  and  $M$  have constant sectional curvature  $-1$ , we can see that

$$h_{ii}h_{jj} - h_{ij}^2 = 0$$

for  $i \neq j$ . Using the fact that  $M$  is maximal, i.e.  $\sum_{i=1}^n h_{ii} = 0$ , we have

$$0 = h_{jj} \sum_{i \neq j} h_{ii} - \sum_{i \neq j} h_{ij}^2 = -h_{jj}^2 - \sum_{i \neq j} h_{ij}^2.$$

it follows that  $M$  is totally geodesic.  $\square$

Now, we are in the position to study the boundary behavior of totally geodesic slice of ADS space  $V$ . For simplicity, we only consider the case that  $\dim V = 4$ .

Let  $\mathbf{R}_2^5$  be 5 dimensional semi-Euclidean space, that is, it is a vector space with the inner product  $\langle X, Y \rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 - x_5y_5$ , where  $X = (x_1, x_2, x_3, x_4, x_5)$  and  $Y = (y_1, y_2, y_3, y_4, y_5)$ . Denote the connection in  $\mathbf{R}_2^5$  by  $\widetilde{\nabla}$ . It is well known that 4 dimensional anti-de Sitter space  $V$  is a totally umbilical hypersurface of  $\mathbf{R}_2^5$ , indeed,  $V = \{X \in \mathbf{R}_2^5 \mid \langle X, X \rangle = -1\}$ .

In the following, we adopt so called sausage coordinate for the anti-de Sitter space  $V$ , namely, any  $X = (x_1, x_2, x_3, x_4, x_5) \in V$  can be expressed by

$$\begin{cases} x_1 &= \frac{2r}{1-r^2} \sin \theta \cos \phi \\ x_2 &= \frac{2r}{1-r^2} \sin \theta \sin \phi \\ x_3 &= \frac{2r}{1-r^2} \cos \theta \\ x_4 &= \frac{1+r^2}{1-r^2} \cos t \\ x_5 &= \frac{1+r^2}{1-r^2} \sin t. \end{cases}$$

where angular coordinates have their usual range, while  $0 \leq r < 1$ . In this coordinates, the Lorentz metric of  $V$  is

$$ds^2 = -\left(\frac{1+r^2}{1-r^2}\right)^2 dt^2 + \frac{4}{(1-r^2)^2} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2),$$

thus,  $t$  can be viewed as a time function in  $V$ . For any slice in  $V$ , we can define its height function by restriction  $t$  on it. Let  $M$  be a slice of  $V$ , then its height function  $u$  can be regarded as a function on  $\mathbf{H}^3$ , which is still denoted by  $u$ . In the sequel, we always assume  $u$  is at least continuous at the infinity boundary of  $\mathbf{H}^3$ , thus, we may define

$$w(\theta, \phi) = \lim_{r \rightarrow 1} u(r, \theta, \phi),$$

hence,  $w$  is a function on  $\mathbf{S}^2$ . Indeed, we have

**Theorem 4.2.** *Suppose  $M$  is a maximal slice in  $V$ , then  $M$  is totally geodesic if and only if there are constants  $w_0, A, B, C$  on  $\mathbf{S}^2$  with  $A^2 + B^2 + C^2 < 1$  such that*

$$(12) \quad f(\theta, \phi) = A \sin \theta \cos \phi + B \sin \theta \sin \phi + C \cos \theta,$$

here  $f = \cos(w + w_0)$ .

**Remark 4.3.** *We would like to point out that  $p = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$  can be regarded as a point on the standard  $\mathbf{S}^2 \subset \mathbf{R}^3$ , and each coordinate component is a first eigenfunction of Laplacian operator on  $\mathbf{S}^2$ .*

*Proof.* Suppose  $M$  is a totally geodesic spacelike silce in  $V$ . Hence  $M$  can also be viewed as a spacelike submanifold in  $\mathbf{R}_2^5$ . We take orthogonal frame  $\{e_1, e_2, e_3, e_4, e_5\}$  for  $\mathbf{R}_2^5$  such that  $e_1, e_2, e_3, e_4$  and  $e_1, e_2, e_3$  are tangent vectors of  $V$  and  $M$  respectively. Denote the position vector of  $V$  by  $X$ . We may assume that  $X = e_5$ . Note that  $M$  is totally geodesic in  $V$  and  $V$  is totally umbilical in  $\mathbf{R}_2^5$ , we get

$$\langle \tilde{\nabla}_{e_i} e_4, e_j \rangle = \langle \tilde{\nabla}_{e_i} e_4, e_5 \rangle = 0,$$

for  $i, j = 1, 2, 3$ , here,  $\tilde{\nabla}$  is the connection in  $\mathbf{R}_2^5$ . Thus, we conclude that  $e_4|_M = a$ , where  $a = (a_1, a_2, a_3, a_4, a_5) \in \mathbf{R}_2^5$  is a constant vector with  $\langle a, a \rangle = -1$ . Furthermore, one have

$$(13) \quad \langle X|_M, a \rangle = \langle X|_M, e_4 \rangle = 0,$$

i.e.,  $M$  is the intersection of  $V$  and a hyperplane  $\Pi_a := \{x \in \mathbf{R}_2^5 \mid \langle x, a \rangle = 0\}$ .

By (13), we obtain

$$\begin{aligned} 0 = \langle X|_M, a \rangle &= a_1 \frac{2r}{1-r^2} \sin \theta \cos \phi + a_2 \frac{2r}{1-r^2} \sin \theta \sin \phi \\ &\quad + a_3 \frac{2r}{1-r^2} \cos \theta - a_4 \frac{1+r^2}{1-r^2} \cos t - a_5 \frac{1+r^2}{1-r^2} \sin t. \end{aligned}$$

Let  $r \rightarrow 1$ , we have

$$\cos(t + w_0) = A \sin \theta \cos \phi + B \sin \theta \sin \phi + C \cos \theta,$$

or equivalently,

$$f(\theta, \phi) = A \sin \theta \cos \phi + B \sin \theta \sin \phi + C \cos \theta,$$

here,  $A = \frac{a_1}{\sqrt{a_4^2 + a_5^2}}, B = \frac{a_2}{\sqrt{a_4^2 + a_5^2}}, C = \frac{a_3}{\sqrt{a_4^2 + a_5^2}}$  and  $\cos w_0 = \frac{a_4}{\sqrt{a_4^2 + a_5^2}}$ .

Conversely, if  $M$  is a maximal slice, and its boundary data satisfies (12), then we choose two constants  $a_4, a_5$  with  $a_4^2 + a_5^2 > 1$  and

$$\begin{aligned}\cos w_0 &= \frac{a_4}{\sqrt{a_4^2 + a_5^2}}, \\ -\sin w_0 &= \frac{a_5}{\sqrt{a_4^2 + a_5^2}}.\end{aligned}$$

Let

$$\begin{aligned}a_1 &= \sqrt{a_4^2 + a_5^2}A \\ a_2 &= \sqrt{a_4^2 + a_5^2}B \\ a_3 &= \sqrt{a_4^2 + a_5^2}C.\end{aligned}$$

Set  $a = (a_1, a_2, a_3, a_4, a_5) \in \mathbf{R}_2^5$  and  $C = \{(1, \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta, w(\theta, \phi)) | 0 < \theta < \pi, 0 \leq \phi \leq 2\pi\}$ , by a direct computation, we see that  $C \subset \Pi_a$ , hence it is the boundary of  $\Pi_a \cap V$  which is totally geodesic slice of  $V$ , in particular, it is maximal slice, therefore, its height function satisfies equation (1), by maximality of  $M$ , we see that the height function of  $M$  also satisfies the same equation, and they are equal at the infinity boundary of  $\mathbf{H}^3$ , thus, by maximal principle, we see they are equal on  $\mathbf{H}^3$  which implies  $M$  is totally geodesic. Thus, we finish to prove the theorem.  $\square$

As a corollary, we have

**Corollary 4.4.** *Let  $M$  be totally geodesic slice in  $V$ , then there is a constant  $w_0$  on  $\mathbf{S}^2$  with*

$$f^2 + |\nabla^{\mathbf{S}^2} f|^2 = C,$$

where  $f = \cos(w + w_0)$ ,  $\nabla^{\mathbf{S}^2}$  is the connection and  $C$  is a constant on  $\mathbf{S}^2$ .

Combine with this fact and Theorem 3.1, we see that the Bernstein Theorem in  $V$  fails.

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